

# HOMOGENEOUS SPECTRUM, DISJOINTNESS OF CONVOLUTIONS, AND MIXING PROPERTIES OF DYNAMICAL SYSTEMS<sup>1</sup>

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## Abstract

In connection with Rokhlin's question on an automorphism with a homogeneous nonsimple spectrum, we indicate a class of measure-preserving maps  $T$  such that  $T \times T$  has a homogeneous spectrum of multiplicity 2. The automorphisms in question satisfy the condition  $\sigma * \sigma \perp \sigma$ , where  $\sigma$  is the spectral measure of  $T$ . We also show that there is a mixing automorphism possessing the above properties and their higher order analogs.

## 1 Introduction

Let  $T$  be an automorphism of a Lebesgue space  $(X, \mu)$ ,  $\mu(X) = 1$ . This automorphism induces a unitary operator  $\hat{T}: L_2(\mu) \rightarrow L_2(\mu)$ ,  $\hat{T}f(x) = f(Tx)$ , and one can speak about spectral measure of the operator  $\hat{T}$  and the multiplicity function of the spectrum.

Rokhlin posed the question on the existence of an automorphism with a nonsimple homogeneous spectrum of finite multiplicity. Automorphisms having nonsimple spectra of finite multiplicity were constructed by Oseledets [8]. Katok [7] obtained the following result: for a generic collection of maps  $T$ , the essential range of the multiplicity function of  $T \times T$  is either  $\mathcal{M}_{T \times T} = \{2\}$  or  $\mathcal{M}_{T \times T} = \{2, 4\}$ . (Goodson and Lemanczyk noted [4] that for  $T \times T$  the spectrum multiplicity function does not take odd values.) Katok conjectured that, for a generic  $T$ , the automorphism  $T \times T$  has a homogeneous spectrum of multiplicity 2. This conjecture was confirmed by Ageev and the author, see [2], [11].

In this paper we study automorphisms  $T$  for which  $\mathcal{M}_{T \times T} = \{2\}$ , i.e.  $T \times T$  has a homogeneous spectrum of multiplicity 2. Namely, we consider automorphisms  $T$  of simple spectrum such that for  $a \in (0, 1)$ , the operator  $(aI + (1 - a)\hat{T})$  (or the operator  $(1 - a)(I + a\hat{T} + a^2\hat{T}^2 + \dots)$ ) belongs to the weak closure of the powers of  $\hat{T}$  (§§2, 3). Using certain staircase constructions [1], one can obtain similar spectral properties for mixing automorphisms (§4). In §5 we discuss disjointness of higher order convolutions.

For definitions, we refer the reader to [3], [12]. We shall denote by  $\sigma = \sigma_T$  the maximal spectral type of the unitary operator  $\hat{T}$  acting on the space  $H = \{f \in L_2(\mu) : \int f d\mu = 0\}$ . It is well known that the spectral type of  $\hat{T} \otimes \hat{T}$  is the measure  $\sigma + \sigma * \sigma$ . Note also that the convolution  $\sigma * \sigma$  is the maximal spectral type of the restriction of  $\hat{T} \otimes \hat{T}$  on the space  $H \otimes H$ .

Throughout we use the fact that the disjointness of  $\sigma$  and  $\sigma * \sigma$  is equivalent to the absence of nonzero operators intertwining  $\hat{T}|_H$  with  $\hat{T} \otimes \hat{T}|_{H \otimes H}$ . We note that the

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conditions

$$(\widehat{T} \otimes \widehat{T})J = J\widehat{T}, \quad J \neq 0$$

imply the existence of a complex measure  $\lambda \neq 0$  such that

$$\widehat{\lambda}(n) = \langle \widehat{T}^n f | J^* g \rangle_{L_2(\mu)} = \langle \widehat{T}^n \otimes \widehat{T}^n Jf | g \rangle_{L_2(\mu \times \mu)},$$

hence  $\lambda$  is a common component of the measures  $\sigma$  and  $\sigma * \sigma$ .

## 2 The case $\widehat{T}^{k_i} \rightarrow (aI + (1-a)\widehat{T})$ .

Goodson [3] noted that, given an automorphism  $T$ , the simplicity of the spectrum of the map  $R: X \times X \rightarrow X \times X$ ,

$$R(x, y) = (y, Tx),$$

implies that  $T \times T$  has a homogeneous spectrum of multiplicity 2. We shall use this in the proofs below.

**Theorem 2.1.** *Let  $T$  be an ergodic automorphism and let the weak convergence*

$$\widehat{T}^{k_i} \rightarrow (aI + (1-a)\widehat{T})$$

*hold for some sequence  $k_i \rightarrow \infty$  and some  $a \in (0, 1)$ . Then*

- (1) *the spectral measure  $\sigma$  of  $\widehat{T}$  and the convolution  $\sigma * \sigma$  are disjoint;*
- (2) *if, in addition,  $T$  has a simple spectrum, then  $(T \times T)$  has a homogeneous spectrum of multiplicity 2.*

Proof. (1) Suppose that a bounded operator  $J$  intertwines  $\widehat{T}$  and  $(\widehat{T} \otimes \widehat{T})$ , i.e.,

$$J\widehat{T} = (\widehat{T} \otimes \widehat{T})J.$$

Setting  $b = 1 - a$  we obtain

$$\begin{aligned} J(aI + b\widehat{T}) &= ((aI + b\widehat{T}) \otimes (aI + b\widehat{T}))J, \\ (a(I \otimes I) + b(\widehat{T} \otimes \widehat{T}))J &= (a^2(I \otimes I) + ab(\widehat{T} \otimes I) + ab(I \otimes \widehat{T}) + b^2(\widehat{T} \otimes \widehat{T}))J, \\ J + (\widehat{T} \otimes \widehat{T})J &= (I \otimes \widehat{T})J + (\widehat{T} \otimes I)J. \end{aligned} \tag{1}$$

This implies that, for all  $i, j$ , we have

$$(\widehat{T}^i \otimes \widehat{T}^j)J + (\widehat{T}^{i+1} \otimes \widehat{T}^{j+1})J - (\widehat{T}^i \otimes \widehat{T}^{j+1})J - (\widehat{T}^{i+1} \otimes \widehat{T}^j)J = 0.$$

Now we obtain

$$\sum_{0 \leq i, j < n} (\widehat{T}^i \otimes \widehat{T}^j)J + (\widehat{T}^{i+1} \otimes \widehat{T}^{j+1})J - (\widehat{T}^i \otimes \widehat{T}^{j+1})J - (\widehat{T}^{i+1} \otimes \widehat{T}^j)J =$$

$$0 = J + (\hat{T}^n \otimes \hat{T}^n)J - (I \otimes \hat{T}^n)J - (\hat{T}^n \otimes I)J. \quad (2)$$

Since  $T$  is weakly mixing (it is not hard to check that an eigenfunction of operator  $\hat{T}$  must be a constant function), we have  $\hat{T}^{n_i} \rightarrow \Theta$ , where  $\Theta$  is the orthogonal projection onto the space of the constant functions. Hence, from (2) we conclude

$$J + (\Theta \otimes \Theta)J = (I \otimes \Theta)J + (\Theta \otimes I)J.$$

Thus,  $\text{Im}(J) \perp H \otimes H$ , which is equivalent to the assertion that the zero operator is a unique operator intertwining  $\hat{T}|_H$  and  $\hat{T} \otimes \hat{T}|_{H \otimes H}$ . It is a well-known fact that the latter is equivalent to

$$\sigma * \sigma \perp \sigma.$$

(2) Let us show that the automorphism  $R$  has a simple spectrum. Let  $f$  be a cyclic vector for the operator  $\hat{T}$  acting on the space

$$H = \left\{ h \in L_2(\mu) : \int h d\mu = 0 \right\}.$$

Let us prove that  $V_{0,0} = f \otimes f$  is a cyclic vector for the restriction of the operator  $\hat{R}$  to the space  $H \otimes H$ . We have to show that all vectors  $V_{m,n} = T^m f \otimes T^n f$  belong to the space  $L$ , the closure of the linear span of the set  $\{R^i V_{0,0} : i \in \mathbf{Z}\}$ . (Note that  $\hat{R}V_{m,n} = V_{n,m+1}$ .) We have

$$\dots V_{0,1}, V_{0,0}, V_{1,1}, \dots \in L.$$

From the relation  $\hat{T}^{k_i} \rightarrow (aI + b\hat{T})$  we obtain

$$[(aI + b\hat{T}) \otimes (aI + b\hat{T})]V_{0,0} = a^2 V_{0,0} + b^2 V_{1,1} + abV_{0,1} + abV_{1,0} \in L.$$

Hence

$$V_{1,0} \in L, \quad V_{0,2} = \hat{R}V_{1,0} \in L.$$

We shall assume that  $V_{0,i} \in L$  for  $i = 0, 1, \dots, p$  and prove that  $V_{0,p+1} \in L$ . It is a well-known fact that the weak closure of the powers  $\hat{T}^n$  is a semigroup, hence it contains the operators  $(aI + b\hat{T})^p$ . It follows that the vector  $U_p = [(aI + b\hat{T})^p \otimes (aI + b\hat{T})^p]V_{0,0}$  belongs to the space  $L$ . We write

$$U_p = (a^p b^p V_{p,0} + a^p b^p V_{0,p}) + \sum_{m,n: |m-n| < p} c_{m,n} V_{m,n},$$

where all  $V_{m,n}$  belong to  $L$  as  $|m - n| < p$ . So we get  $(V_{p,0} + V_{0,p}) \in L$ . This implies that

$$V_{p,0} \in L, \quad V_{0,p+1} = \hat{R}V_{p,0} \in L.$$

Thus we have proved that, for all  $m, n$ ,  $V_{m,n} = R^{2m} V_{0,n-m} \in L$ , i.e.,  $L = H \otimes H$ , the restriction of  $\hat{R}$  to  $H \otimes H$ , has a simple spectrum. Note that the restriction of  $\hat{R}$  to  $(1 \otimes H) + (H \otimes 1)$  has also a simple spectrum ( $1 \otimes f$  is a cyclic vector). Since the action

of the operator  $\widehat{R}^2$  on  $(1 \otimes H) + (H \otimes 1)$  and the action of  $\widehat{R}^2$  on  $(H \otimes H)$  are disjoint, the same is true for  $\widehat{R}$  and we obtain that  $R$  has a simple spectrum.

**Remarks.** (i) Theorem 2.1 was proved in part independently by Ageev, see [2]. He also gave a solution to Katok's conjecture concerning the sets of spectral multiplicities of  $T \times T \times \dots \times T$  for generic automorphisms  $T$ .

(ii) Katok and Stepin pointed out that there is a three interval exchange transformation  $T$  (in fact an automorphism of the half-circle, induced by some rotation of the unit circle) possessing the property of the  $(n, n+1)$ -type approximation (see [7], [5] for the definition). This property implies, for some sequence  $h(i) \rightarrow \infty$  and any  $p > 0$ , the weak convergence

$$T^{ph(i)} \rightarrow \frac{1}{2}(I + T^p). \quad (*)$$

Thus, we have interval exchange transformations satisfying the conditions of Theorem 2.1. In addition, we obtain  $\kappa$ -mixing property for our  $T$  for  $\kappa = \frac{1}{2}$ , i.e., for some sequence  $k_i \rightarrow \infty$ , one has

$$\widehat{T}^{k_i} \rightarrow \frac{1}{2}(I + \Theta), \quad (**)$$

where  $\Theta$  is the orthogonal projection onto the space of the constant functions. This confirms the corresponding *Oseledets conjecture* from [9] (see also [3]). To see that  $(*)$  implies  $(**)$  we use the fact that, for some sequence  $p_j \rightarrow \infty$ , we have  $\widehat{T}^{p_j} \rightarrow \Theta$ , since  $T$  is weakly mixing.

### 3 The case $\widehat{T}^{k_i} \rightarrow (1 - a)(I + a\widehat{T} + a^2\widehat{T}^2 + \dots)$ .

In this section we consider automorphisms which can be close to the class of mixing automorphisms: if  $a$  is close to 1, then, for ergodic  $T$ , the operator  $(1 - a)(I + a\widehat{T} + a^2\widehat{T}^2 + \dots)$  will be close to  $\Theta$ .

**Theorem 3.1.** *Suppose that  $T$  is an ergodic automorphism such that, for some sequence  $k_i \rightarrow \infty$  and some  $a \in (0, 1)$ , one has the weak convergence*

$$\widehat{T}^{k_i} \rightarrow (1 - a)(I + a\widehat{T} + a^2\widehat{T}^2 + \dots).$$

Then

- (1) *the spectral measure  $\sigma$  of  $\widehat{T}$  and the convolution  $\sigma * \sigma$  are disjoint;*
- (2) *if  $T$  has a simple spectrum, then the automorphism  $(T \times T)$  has a homogeneous spectrum of multiplicity 2.*

Proof. (1) We denote

$$P = (1 - a)(I - a\widehat{T})^{-1} = (1 - a)(I + a\widehat{T} + a^2\widehat{T}^2 + \dots).$$

Let an operator  $J: H \rightarrow H \otimes H$  satisfy the condition

$$J\widehat{T} = (\widehat{T} \otimes \widehat{T})J.$$

Since  $\hat{T}^{k_i} \rightarrow P$ , we have

$$\begin{aligned} JP &= (P \otimes P)J, \\ J(1-a)(I + a\hat{T} + a^2\hat{T}^2 + \dots) &= (P \otimes P)J, \\ (1-a)[I \otimes I + a(\hat{T} \otimes \hat{T}) + a^2(\hat{T} \otimes \hat{T})^2 + \dots]J &= (P \otimes P)J, \\ (1-a)(I \otimes I - a(\hat{T} \otimes \hat{T}))^{-1}J &= (1-a)^2(I - a\hat{T})^{-1} \otimes (I - a\hat{T})^{-1}J. \end{aligned}$$

For the commuting operators  $A = (I \otimes I - a(\hat{T} \otimes \hat{T}))$  and  $B = (I - a\hat{T}) \otimes (I - a\hat{T})$ , the equality  $A^{-1}J = B^{-1}J$  implies  $AJ = BJ$ , hence we obtain

$$\begin{aligned} (1-a)(I \otimes I - a(\hat{T} \otimes \hat{T}))J &= (I - a\hat{T}) \otimes (I - a\hat{T})J, \\ [I \otimes I + \hat{T} \otimes \hat{T}]J &= [I \otimes \hat{T} + \hat{T} \otimes I]J. \end{aligned}$$

As it was proved above, the latter implies  $J = 0$ , which is equivalent to  $\sigma * \sigma \perp \sigma$ .

(2) As in the proof of Theorem 2.1, we show that the space  $H \otimes H$  is a cyclic space for the operator  $\hat{R}$ . Let us use the following well-known fact: if there is a sequence of cyclic spaces

$$C_0 \subseteq C_1 \subseteq C_2 \subseteq C_3 \subseteq \dots,$$

whose union is dense in  $H \otimes H$ , then  $H \otimes H$  is a cyclic space. Let  $C_n$ ,  $n = 0, 1, 2, \dots$ , be the cyclic space (for the operator  $\hat{R}$ ) generated by the vector  $W_n$ , where

$$W_n = [(I - a\hat{T})^n \otimes (I - a\hat{T})^n]f \otimes f$$

( $f$  is a cyclic vector for  $\hat{T}$  on  $H$ ).

Since  $C_n$  is invariant with respect to  $\hat{R}^2 = \hat{T} \otimes \hat{T}$ , from  $\hat{T}^{k(i)} \rightarrow P$  we obtain  $(P \otimes P)C_n \subseteq C_n$ . But  $(P \otimes P)W_n = W_{n-1}$ , thus,  $C_{n-1} \subseteq C_n$ .

Let  $L$  be the closure of the union of the  $C_n$ 's. We have to show that all vectors  $V_{n,0}$  belong to  $L$  (this implies that all  $V_{m,n} \in L$ ).

Let us check that  $V_{1,0} \in L$ . We have

$$V_{0,1}, V_{0,0}, V_{1,1}, W_1 = [(I - a\hat{T}) \otimes (I - a\hat{T})]V_{0,0} \in C_1.$$

Since  $W_1 = V_{0,0} + a^2V_{1,1} - aV_{0,1} - aV_{1,0}$ , we get  $V_{1,0} \in C_1$ .

We obtain  $\hat{R}V_{1,0} = V_{0,2} \in C_1$ . Let us prove that  $V_{2,0} \in C_2$ . The vector  $W_2$  is a linear combination of the vectors  $V_{i,j}$ ,  $0 \leq i, j \leq 2$ . We know that all these vectors except  $V_{2,0}$  are in  $C_1$ . But  $W_2 \in C_2$  and  $C_1 \subseteq C_2$ , hence  $V_{2,0} \in C_2$ .

By induction, we obtain

$$V_{n,0} \in C_n \subseteq L, \quad \forall n = 0, 1, 2, \dots$$

As in the proof of Theorem 2.1 we conclude that  $L = H \otimes H$  and obtain the simplicity of the spectrum of the operator  $\hat{R}$ .

Theorems 2.1 and 3.2 have been announced in [11].

## 4 The case of mixing $T$ .

A generalization of the above methods enables us to prove the existence of a mixing operator  $T$  with the following properties:

1. Spectrum of the symmetric product  $T \odot T$  is simple:  $\mathcal{M}_{T \odot T} = \{1\}$ ,
2.  $\mathcal{M}_{T \times T} = \{2\}$ ,
3.  $\sigma_T * \sigma_T \perp \sigma_T$ .

We recall that  $T \odot T$  denotes the restriction of  $T \times T$  to the factor of  $S$ -fixed subsets of  $X \times X$ , where the map  $S$  is defined as  $S(x, y) = (y, x)$ . Note also that Property 1 implies Property 2 and Property 3, which is readily seen.

Adams [1] proved the property of mixing for a large class of rank 1 staircase constructions. For some special automorphisms of the Adams class, we are able to prove Property 1. This is a positive answer to the corresponding question of J.-P. Thouvenot.

Let us recall the definition of a staircase construction. Let an automorphism  $T$  admit, for any  $n$ , a partition  $\xi_n$  of  $X$  into sets

$$\begin{aligned} & B_n^1, TB_n^1, \dots, T^{h_n-1}B_n^1, \\ & B_n^2, TB_n^2, \dots, T^{h_n-1}B_n^2, T^{h_n}B_n^2, \\ & B_n^3, TB_n^3, \dots, T^{h_n-1}B_n^3, T^{h_n}B_n^3, T^{h_n+1}B_n^3, \\ & \dots \dots \dots \\ & B_n^{r_n}, TB_n^{r_n}, \dots, T^{h_n+1}B_n^{r_n}, T^{h_n+2}B_n^{r_n}, \dots, T^{h_n+r_n-2}B_n^{r_n}, Y_n \end{aligned}$$

such that

$$B_n^2 = T^{h_n}B_n^1, B_n^3 = T^{h_n+1}B_n^2, \dots, B_n^{r_n} = T^{h_n+r_n-1}B_n^{r_n-1}$$

for all  $n$ , and the sequence of the partitions  $\xi_n$  tends to the partitions into singletons ( $\xi_n \rightarrow \varepsilon$ ). If, in addition, for all  $n$  we have

$$B_{n-1}^1 = B_n^1 \sqcup B_n^2 \sqcup \dots \sqcup B_n^{r_n},$$

we say that such an automorphism  $T$  is a *staircase construction*. One can see that this construction is defined by  $h_1$  and the sequence  $\{r_n\}$ .

It is easily seen that  $h_n + 1$  is the number of atoms in the partition  $\xi_{n-1}$ . Note also that the set  $Y_{n-1}$  is the union of the sets

$$T^{h_n}B_n^2, T^{h_n}B_n^3, T^{h_n+1}B_n^3, \dots, T^{h_n+2}B_n^{r_n}, \dots, T^{h_n+r_n-2}B_n^{r_n}, Y_n.$$

The Adams theorem asserts that in the case where  $r_n \rightarrow \infty$  and  $\frac{(r_n)^2}{h_n} \rightarrow 0$ , the corresponding staircase construction  $T$  is mixing.

We can prove that there is a sequence  $r_n \rightarrow \infty$  such that  $\frac{(r_n)^2}{h_n} \rightarrow 0$ , and the corresponding operator  $T$  possesses the property  $\mathcal{M}_{T \odot T} = \{1\}$ .

**Theorem 4.1.** *There is a staircase mixing construction  $T$  possessing the property  $\mathcal{M}_{T \odot T} = \{1\}$ .*

We introduce special classes  $St.C.(p_j, h_j)$  of staircase constructions, and prove that the property  $\mathcal{M}_{T_j \odot T_j} = \{1\}$  holds for any  $T_j \in St.C.(p_j, h_j)$ . Then we find a sequence of corresponding  $T_j$  which approximates sufficiently well some staircase construction  $T$  satisfying the conditions of the Adams theorem. Due to the approximation procedure, the property  $\mathcal{M}_{T \odot T} = \{1\}$  will be preserved.

*Definition.* We say that  $T$  is in the class  $St.C.(p, h)$  if  $T$  is a staircase construction with a sequence  $r_n$  satisfying the following conditions:

1.  $\liminf_{n \rightarrow \infty} r_n = p$ ;
2.  $\forall q, p \leq q \leq h, \exists n_i \rightarrow \infty$  such that  $r_{n_i+1} \rightarrow \infty$  and  $\forall i, r_{n_i} = q$ .

It follows from Condition 2 that the operators

$$P_q = \frac{1}{q}(I + \hat{T} + \hat{T}^2 + \dots + \hat{T}^{q-2} + \Theta)$$

are in  $WCl(T)$  as  $p \leq q \leq h$ . (In fact the powers  $\hat{T}^{-h_{n_i}}$  converge weakly to  $P_q$  as  $r_{n_i} = q$  for all  $i$ .)

**Lemma.** Let  $T$  be in  $St.C.(p, h+2)$ , and let  $3p < h+2$ . Suppose that  $B, TB, \dots, T^h B$  are disjoint measurable sets, and denote by  $C_{B \times B}$  the cyclic space generated by vector  $\chi_B \otimes \chi_B$  under the action of the operator  $\hat{T} \otimes \hat{T}$ . Then the functions

$$F_{i,j} = \chi_{T^i B} \otimes \chi_{T^j B} + \chi_{T^j B} \otimes \chi_{T^i B}, \quad 0 \leq i, j \leq h,$$

belong to the cyclic space  $C_{B \times B}$ .

Proof. Since

$$F_{q,0} \in C_{B \times B} \implies F_{q+i,i} \in C_{B \times B}$$

we only show that  $F_{q,0} \in C_{B \times B}$ . We have

$$P_{q+2}\chi_B \otimes P_{q+2}\chi_B, \quad P_{q+1}\chi_B \otimes P_{q+1}\chi_B, \quad P_q\chi_B \otimes P_q\chi_B \in C_{B \times B}.$$

Let

$$G_m = \frac{m^2}{(1 + \mu(B))^2} P_m \chi_B \otimes P_m \chi_B.$$

One can check that

$$F_{q,0} = \text{Const} [G_{q+2} - G_{q+1} - (\hat{T} \otimes \hat{T})G_{q+1} + (\hat{T} \otimes \hat{T})G_q].$$

Thus, for all  $q \geq p$ , we have  $F_{q,0} \in C_{B \times B}$ , hence, all the functions  $F_{i,j}$  are in  $C_{B \times B}$ .

Now we prove that, for all  $q < p$ , we have  $F_{q,0} \in C_{B \times B}$  too. Let us show this for  $q = 1$ . Since

$$F_{p,0} = [(\hat{T}^p \otimes I) + (I \otimes \hat{T}^p)]\chi_{B \times B}$$

and  $C_{B \times B}$  contains  $F_{p+1,0}$ , we obtain that the cyclic space generated by  $F_{p,0}$  contains

$$[(\hat{T}^p \otimes I) + (I \otimes \hat{T}^p)]F_{p+1,0} = [(\hat{T}^p \otimes I) + (I \otimes \hat{T}^p)][(\hat{T}^{p+1} \otimes I) + (I \otimes \hat{T}^{p+1})]\chi_{B \times B}.$$

The latter can be represented as the sum  $F_{2p+1,0} + F_{p+1,p}$ , where  $F_{2p+1,0} \in C_{B \times B}$ . Since this sum is also in  $C_{B \times B}$ , we get  $F_{p+1,p} \in C_{B \times B}$  and, hence  $F_{1,0}$  belongs to  $C_{B \times B}$ .

**Theorem 4.2.** *Let  $T$  belong to  $St.C.(p, \infty)$ . Then  $\mathcal{M}_{T \odot T} = \{1\}$ .*

Proof. We consider the sequence  $\xi_n$  and the corresponding sequence of the sets  $B_n = B_n^1$  (see the above definition of a staircase construction). Let us consider the sequence of the cyclic spaces  $C_{B_n \times B_n}$  for the operator  $\hat{T} \otimes \hat{T}$ . Each symmetric function  $F(x, y)$  can be approximated by linear combinations of  $F_{i,j}$ , where the functions  $F_{i,j}$  depend on  $n$  and  $0 \leq i, j \leq h_n$ . From the above lemma we obtain that the symmetric product  $L_2(\mu) \odot L_2(\mu)$  can be approximated by the cyclic spaces  $C_{B_n \times B_n}$ . The Katok–Oseledets–Stepin approach yields that  $L_2(\mu) \odot L_2(\mu)$  is a cyclic space as well. Thus  $\mathcal{M}_{T \odot T} = \{1\}$ .

How to construct a mixing operator  $T$  with the property  $\mathcal{M}_{T \odot T} = \{1\}$ ? In fact we prove only the existence. A non-constructive description is this: we consider a sequence  $\{r_n\}$  such that, for all  $k$ , we have  $r_{2k+1} = 2k + 1$ , and the sequence  $r_{2k}$  *extremely slowly* tends to the infinity as  $k \rightarrow \infty$ . Now we explain why such a sequence  $\{r_n\}$  can give the desired staircase construction.

We shall consider a sequence of automorphisms  $T_j$  such that  $T_j$  is of class  $St.C(p_j, \infty)$  on  $X_j = \{x : T_j(x) \neq x\}$ , where  $\mu(X_j) \rightarrow 1$  and  $p_j \rightarrow \infty$ . Given an automorphism  $T_j$  with the corresponding sequence  $r_n^{(j)}$ , we choose a sufficiently large number  $N_j$  and change this sequence only for  $n > N_j$ . We obtain a new sequence  $r_n^{(j+1)}$  and the corresponding construction  $T_{j+1}$  such that  $T_j$  differs from  $T_{j+1}$  only on the very small set  $Y_{N_j}$  (here  $Y_{N_j}$  corresponds to the automorphism  $T_{j+1}$ ). Our operator  $T$  will be a limit automorphism, in fact a staircase construction with  $r_n \rightarrow \infty$ . We can choose  $r_n \leq n$ , then we obtain  $\frac{(r_n)^2}{h_n} \rightarrow 0$ , which guarantees the property of mixing.

The sequence  $\{T_j\}$  is organized so that, for any symmetric functions  $F(x, y)$ , the distance between  $F$  and the space  $C_{B_j \times B_j}$  (here the cyclic space is considered for the operator  $\hat{T}_j \otimes \hat{T}_j$ ) tends to zero. However, it is possible to obtain the same property for the sequence of the spaces  $C'_{B_j \times B_j}$ , the cyclic spaces of the operator  $\hat{T} \otimes \hat{T}$ . Indeed, let for some  $F$  ( $F$  will be taken from a fixed finite collection of symmetric functions) we have

$$\left\| F - \sum_{k=-N}^N a_k U_j^k \chi_{B_j \times B_j} \right\| < \varepsilon,$$

where  $U_j = \hat{T}_j \otimes \hat{T}_j$ . However, we can ensure that

$$\left\| F - \sum_{k=-N}^N a_k U^k \chi_{B_j \times B_j} \right\| < 2\varepsilon$$

for  $U = \hat{T} \otimes \hat{T}$ , since the measure of the set  $\{x : T_j(x) \neq T(x)\}$  can be made as small as desired by an appropriate choice of  $T_{j+1}, T_{j+2}, \dots$ .

Thus, it is possible to find a construction such that, for any fixed countable set of symmetric functions  $F$  and, hence, for all symmetric functions (due to the separability of  $L_2 \otimes L_2$ ), the distance between  $F$  and the space  $C'_{B_j \times B_j}$  tends to 0 as  $j \rightarrow \infty$ .



Thus we conserve the property  $\mathcal{M}_{T \odot T} = \{1\}$  for some mixing staircase construction  $T$ .

**Remark.** It is worth noting that for mixing  $T$  the property

$$\mathcal{M}_{T \odot T} = \{1\}$$

implies mixing of all orders. This follows from Host's theorem: if  $\sigma_T * \sigma_T \perp \sigma_T$ , then the mixing automorphism  $T$  possesses the multiple mixing property [6].

## 5 On higher order properties.

Disjointness of  $\sigma * \sigma$  and  $\sigma$  in the case  $2\hat{T}^{k_i} \rightarrow (I + \hat{T})$  was obtained also by M. Lemanczyk and generalized by F. Parreau to  $\sigma^{*n} \perp \sigma$ . The problem " $\sigma^{*n} \perp \sigma^{*m}$ ?" as  $m > n > 1$  remains open. In some cases, we can obtain a bit more than  $\sigma^{*n} \perp \sigma$ .

**Theorem 5.1.**

*If, for an ergodic automorphism  $T$ , there is a sequence  $k_i \rightarrow \infty$  such that*

$$\hat{T}^{k_i} \rightarrow P = (aI + b\hat{T}), \quad 1 > a > b > 0,$$

*then  $\sigma * \sigma \perp \sigma * \sigma * \sigma$ , where  $\sigma$  is the spectral measure of the automorphism  $T$ .*

Proof. Suppose that some operator  $J: L_2 \otimes L_2 \rightarrow L_2 \otimes L_2 \otimes L_2$  satisfies the intertwining condition

$$J(\hat{T} \otimes \hat{T}) = (\hat{T} \otimes \hat{T} \otimes \hat{T})J.$$

We have to show that  $J = 0$ . Since

$$\hat{T}^{-k_i+1} \rightarrow Q = (bI + a\hat{T}),$$

we obtain

$$J[P \otimes P - Q \otimes Q] = [P \otimes P \otimes P - Q \otimes Q \otimes Q]J, \quad J(a^2 - b^2)[I \otimes I - \hat{T} \otimes \hat{T}] = [P \otimes P \otimes P - Q \otimes Q \otimes Q]J,$$

$$J(a^2 - b^2) = [P \otimes P \otimes P - Q \otimes Q \otimes Q + (a^2 - b^2)\hat{T} \otimes \hat{T} \otimes \hat{T}]J,$$

$$(a^2 - b^2 - a^3 + b^3)J = [\dots + (a^2 - b^2 - a^3 + b^3)\hat{T} \otimes \hat{T} \otimes \hat{T}]J,$$

$$J = [(\dots) + \hat{T} \otimes \hat{T} \otimes \hat{T}]J,$$

where  $(\dots)$  is a linear combination of the operators  $\hat{T} \otimes I \otimes I, \dots, I \otimes \hat{T} \otimes \hat{T}$ . Let, for  $h \in H \otimes H$ , we have

$$Jh = [(\dots) + \hat{T} \otimes \hat{T} \otimes \hat{T}]Jh.$$

We rewrite this for the spectral representation of  $\hat{T}$  as follows:

$$f(x, y, z) = [(\dots) + xyz]f(x, y, z),$$

where  $f$  is the image of  $Jh$  in the spectral representation, and  $x, y, z$  belong to the unit circle. We can see that, for any fixed  $x, y$ , there is a unique point  $z$  such that

$$0 \neq f(x, y, z) = [(\dots) + xyz]f(x, y, z).$$

Since the measure  $\sigma$  is continuous, one has  $\sigma \otimes \sigma \otimes \sigma(\text{support}(f)) = 0$ , i.e.,  $f = 0$  in the space  $L_2(\sigma \otimes \sigma \otimes \sigma)$ . Thus we obtain

$$Jh = 0, \quad J = 0, \quad \sigma * \sigma \perp \sigma * \sigma * \sigma.$$

The following assertion is a natural generalization of Theorem 4.1.

**Theorem 5.2.** *There is a mixing automorphism  $T$  with the following properties:*

- (1)  $\mathcal{M}_{T^{\otimes n}} = \{1\}$ ,
- (2)  $\mathcal{M}_{T^{\times n}} = \{n, n(n-1), \dots, n!\}$ ,
- (3)  $\sigma^{*k} \perp \sigma^{*m}$  for all  $k > m > 0$ .

The proof of Theorem 5.2 will be published in a separate paper. <sup>2</sup>

**Remark.** Ageev [2] proved property (2) for generic (non-mixing) automorphisms. In [10], property (3) has been established for the well-known Chacon automorphism. We conjecture that the Chacon automorphism has properties (1) and (2) as well.

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<sup>2</sup>See V.V. Ryzhikov, Weak limits of powers, simple spectrum of symmetric products, and rank-one mixing constructions, Sbornik: Mathematics (2007), 198(5):733–754

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